$|f(t_k) - f(c)| < \epsilon$

Hence proof f is continuous at c,

$\forall P$ such that $\|P\| < \delta$,

(2) $\Rightarrow |\Delta| \leq \epsilon\, |\alpha(c) - \alpha(c-)| + \epsilon\, |\alpha(c+) - \alpha(c)| \rightarrow (6)$

Case: (i)

If f is discontinuous both from left ~~and~~ at c and from the right at c, then.

By hypothesis, α is continuous from the left at c and from the right at c

Let $\alpha(c+) = \alpha(c)$ & $\alpha(c-) = \alpha(c)$

Then (1) $\Rightarrow \Delta = 0$

$\therefore |\Delta| = 0 \quad |S(P, f, \alpha) - A| < \epsilon$

Case: (iii)

If f is continuous from the left at c and f is not continuous from the right at c

then α is continuous from the right at c and α is not continuous from the left at c.

Then

$\alpha(c+) = \alpha(c)$

$\Rightarrow \alpha(c+) - \alpha(c) = 0 \Rightarrow |\alpha(c+) - \alpha(c)| = 0$

in this case,

$(2) \Rightarrow |\Delta| = |f(t_{k-1}) - f(c)|\, |\alpha(c) - \alpha(c-)|$

$\Rightarrow |\Delta| \leq \epsilon\, |\alpha(c) - \alpha(c-)|$ $\Big\{ \because f$ is continuous from the left at c, $\forall \epsilon > 0\, \exists\, \delta > 0$ such that $\Rightarrow |f(t_{k-1}) - f(c)| < \epsilon$, $|t_{k-1} - c| < \delta \Big\}$

Case: (iv)

Similarly, if f is continuous from the right at c and f is not continuous from the left at c, then α is continuous from the left at c and α is not continuous from the right at c.

Then $\alpha(c-) = \alpha(c) \Rightarrow |\alpha(c-) - \alpha(c)| = 0$

$\therefore$ (2) $\Rightarrow |A| = |f(t_k) - f(c)|\,|\alpha(c+) - \alpha(c)|$.

$\Rightarrow |A| < \epsilon\, |\alpha(c+) - \alpha(c)|$

(3) is valid in this case also.

f is continuous from the right at c. $|f(t_k) - f(c)| < \epsilon$ if $|t_k - c| < \delta$

If we choose P_ϵ of $[a,b]$ such that $\|P_\epsilon\| < \delta$

then $P \supseteq P_\epsilon \Rightarrow \|P\| \leq \|P_\epsilon\| < \delta$

$\therefore$ for such P, (3) is valid

$\therefore$ (for any given $\epsilon > 0$, $\exists$ a partition P_ϵ such that $\forall$ $P \supseteq P_\epsilon$ implies)

$$|A| < \epsilon\,|\alpha(c) - \alpha(c-)| + \epsilon\,|\alpha(c+) - \alpha(c)|$$

$$\therefore |S(P,f,\alpha) - A| \leq \epsilon\,|\alpha(c) - \alpha(c-)| + \epsilon\,|\alpha(c+) - \alpha(c)|$$

$$= \epsilon\,[\,|\alpha(c) - \alpha(c-)| + |\alpha(c+) - \alpha(c)|\,]$$

$\Rightarrow f \in R(\alpha)$ on $[a,b]$ and $A = \int_a^b f\,d\alpha$

$\Rightarrow f \in R(\alpha)$ on $[a,b]$ and

$$f(c)\,[\alpha(c+) - \alpha(c-)] = \int_a^b f\,d\alpha$$

Hence the proof.

Note:

If $c = a$, then also the theorem is valid and in this case,

$$\int_a^b f\,d\alpha = f(a)\,[\alpha(a+) - \alpha(a)]$$

$\because$ a is the left end point $\Rightarrow a- = a \Rightarrow \alpha(a-) = \alpha(a)$

and if $c = b$ we have

$$\int_a^b f\,d\alpha = f(b)\,[\alpha(b) - \alpha(b-)]$$

$\because$ b is right end point $\Rightarrow b+ = b \Rightarrow \alpha(b+) = \alpha(b)$

Example:

The existence of a Riemann - Stieltjes integral may be affected if the value of the integrand is altered at a single point.

Proof: Using a counter example

Define α and f on $[-1,1]$ as follows;

$\alpha(x) = 0$ if $x \neq 0$

$\alpha(0) = -1$

and $f(x) = 1$ if $-1 \leq x \leq 1$

Then, By theorem 7:

$\int_{-1}^{1} f\,d\alpha$ exists and ie) $f \in R(\alpha)$ on $[-1,1]$

$$\int_{-1}^{1} f\,d\alpha = f(0)\left[\alpha(0+) - \alpha(0-)\right]$$

$$= 1[0-0]$$

$$= 0$$

But if we redefine f, so that

$f(0) = 2$ and $f(x) = 1$ if $x \neq 0$

then $\int_{-1}^{1} f\,d\alpha$ does not exist.

Soln: Another method

Let $P = \{-1 = x_0, x_1, \ldots x_{k-2}, 0 = x_{k-1}, x_k \ldots x_n = 1\}$

be a partition of $[-1,1]$

Since α is a constant on $[-1, x_{k-2}]$ and on $[x_{k-1}, x_n]$ we have,

$$S(P, f, \alpha) = f(t_k)\left[\alpha(x_k) - \alpha(0)\right] + f(t_{k-1})\left[\alpha(0) - \alpha(x_{k-2})\right]$$

$$= f(t_k)\left[0 - (-1)\right] + f(t_{k-1})\left[-1 - 0\right]$$

$$= f(t_k) - f(t_{k-1})$$

where $x_{k-1} \le t_{k-1} \le 0 \le t_k \le x_k$

If $t_k = t_{k-1} = 0$ then $S(P, f, \alpha) = 0$

If $t_k = 0$, $t_{k-1} \neq 0$ then $S(P, f, \alpha) = 1$

If $t_{k-1} = 0$, $t_k \neq 0$ then $S(P, f, \alpha) = -1$

Hence $\int_{-1}^{1} f d\alpha$ does not exist.

7.9 Reduction of a Riemann Stieltjes Integral to a Finite Sum

Definition:

Step Function:

A function α defined on $[a, b]$ is called a step function if there is a partition,

$a = x_1 < x_2 < \dots < x_n = b$ such that α is constant on each open subinterval (x_{k-1}, x_k)

The number $\alpha(x_k+) - \alpha(x_k-)$ is called the Jump at x_k if $1 < k < n$

The Jump at x_1 is $\alpha(x_1+) - \alpha(x_1-)$ and the Jump at x_n is $\alpha(x_n+) - \alpha(x_n-)$

Example for step function:

The greatest integer function denoted by $[x]$ is defined as unique integer satisfying the inequalities $[x] \le x < [x] + 1$

ie) $[2.001] = 2$, $[2.1] = 2$, $[2.9999] = 2$, $[2] = 2$

Claim:

$\alpha(x) = [x]$ is a step function on $[a, b]$ where n is a positive integer.

Consider the partition,

$\{0, 1, 2, \dots, n\}$ of $[0, n]$

In each open subinterval $[k-1, k)$,

$\alpha(x) = k-1, \ \forall \ x \in [k-1, k)$

$\therefore$ α is a constant in each of the open interval,

$\therefore$ α is a step function on $[0, n]$

The jump of α at k,

$= \alpha(k+) - \alpha(k-)$ if $0 < k < n$

$= k - (k-1)$

$= 1 \qquad \forall \ 0 < k < n$

The jump of α at 0 $= \alpha(0+) - \alpha(0)$

$= 0 - 0 = 0 \qquad \boxed{\alpha_0 = 0}$

The jump of α at n $= \alpha(n) - \alpha(n-1)$

$= n - (n-1)$

$= 1.$

$\alpha_n = \alpha(n^+) - \alpha(n^-)$
$= n - (n-1)$
$= n - n + 1$
$= 1$

Thus The jump of α at k is 1, for $k = 1, 2, \dots n$

The jump of α at 0 is 0.

Theorem : 8

Reduction of a Riemann - Stieltjes Integral to a finite Sum:

Let α be a step function defined on $[a, b]$ with jump α_k at x_k, where $x_1, x_2, \dots x_n$ are such that $x_1 < x_2 < \dots < x_n$

Let f be defined on $[a, b]$ in such a way that not both 'f' and 'α' are discontinuous from the right or from the left at each x_k

Then $\int_a^b f\,d\alpha$ exists and we have

$$\int_a^b f(x)\,d\alpha(x) = \sum_{k=1}^{n} f(x_k)\cdot \alpha_k$$

Proof:

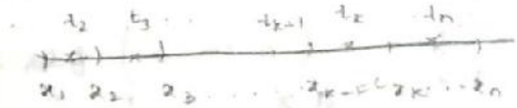
Let α be a step function defined on $[a,b]$.

Let $P=\{x_1, x_2 \dots x_n\}$ be a partition of $[a,b]$

Such that,

α is a constant in each one of the open

Subintervals (x_{k-1}, x_k)



Let $t_k \in (x_{k-1}, x_k)$, $k = 2, 3, \dots n$

consider the intervals $[t_{k-1}, t_k]$, $k = 3, 4, \dots n$

then, $\alpha(x) = \alpha(t_{k-1})$ when $t_{k-1} \le x \le x_{k-1}$

and $\alpha(x) = \alpha(t_k)$ when $x_{k-1} \le x \le t_k$

By hypothesis atleast one of f and α is continuous from the left at each x_k and atleast one is continuous from the right at each x_k.

$\therefore$ By theorem 7,

$\int_{t_{k-1}}^{t_k} f\,d\alpha$ exists and

$\int_a^b f\,d\alpha = f(c)\,[\alpha(c+) - \alpha(c-)]$ $\quad c = x_{k-1}$

$$\int_{t_{k-1}}^{t_k} f\,d\alpha = f(x_{k-1})\left[\alpha(x_{k-1}^{+}) - \alpha(x_{k-1}^{-})\right]$$

$$= f(x_{k-1})\,\alpha_{k-1} \quad \text{for } k = 3, 4, \dots n$$

$\int_{a=x_1}^{t_2} f\,d\alpha$ exists and

$$\int_{a=x_1}^{t_2} f\,d\alpha = f(x_1)\,[\alpha(x_1+) - \alpha(x_1)]$$

$$= f(x_1)\cdot\alpha_1$$

Similarly, $\int_{t_n}^{b=x_n} f\,d\alpha$ exists and

$$\int_{t_n}^{b=x_n} f\,d\alpha = f(x_n)\left[\alpha(x_n) - \alpha(x_n-)\right]$$

$$= f(x_n)\cdot \alpha_n$$

$\therefore$ By thm (3)

$\int_a^b f\,d\alpha$ exists and

$$\int_{a=x_1}^{b=x_n} f\,d\alpha = \int_{a=x_1}^{t_2} f\,d\alpha + \int_{t_2}^{t_3} f\,d\alpha + \cdots + \int_{t_{n-1}}^{t_n} f\,d\alpha + \int_{t_n}^{x_n} f\,d\alpha$$

$$= f(x_1)\alpha_1 + f(x_2)\alpha_2 + \cdots + f(x_{n-1})\alpha_{n-1} + f(x_n)\alpha_n$$

$$= \sum_{k=1}^{n} f(x_k)\alpha_k \quad \text{where } \alpha_k \text{ is the jump of } \alpha \text{ at } x_k.$$

Hence proved the result

we shall also proceed this way

Consider the intervals $[t_k, t_{k+1}]$, $k = 2, 3, \ldots n-1$

then $\alpha(x) = \alpha(t_k)$, if $t_k \le x < x_k$

$\alpha(x) = \alpha(t_{k+1})$, if $x_k < x \le t_{k+1}$

$\therefore$ By theorem 7,

$\int_{t_k}^{t_{k+1}} f\,d\alpha$ exists and

$$\int_{t_k}^{t_{k+1}} f\,d\alpha = f(x_k)\left[\alpha(x_k+) - \alpha(x_k-)\right] = f(x_k)\alpha_k$$

$$= f(x_k)\,[\text{jump of } \alpha \text{ at } k]$$

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Theorem: 9

Every finite sum can be written as a Riemann Stieltjes Integral. In fact given a sum, $\sum_{k=1}^{n} a_k$, define 'f' on $[0, n]$ as follows,

$$f(x) = a_k \quad \text{if} \quad k-1 < x \leq k, \ (k = 1, 2, \dots n)$$

$$f(0) = 0$$

Then $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} f(k) = \int_0^n f(x)\, d[x]$

where $[x]$ is the greatest integer function less than or equal to 'x' ie $[x] \leq x$.

Proof: Let $\alpha(x) = [x]$. be the greatest integer function defined on $[0, n]$

Then α is a step function on $[0, n]$

Also, the jump of α at 0 is 0 and the jump of α at $k = 1, 2, 3 \dots n$ is 1.

Now $\alpha(k+) = [k+] = k$

and $\alpha(k) = [k] = k$

$\therefore \alpha(k+) = \alpha(k)$

Hence α is continuous from the right of k.

Now $f(k-) = a_k$ and $f(k) = a_k$

$\Rightarrow f(k-)$ and $f(k)$

Hence f is continuous from the left of k.

Therefore all the conditions of theorem: 8 is satisfied by f and α.

And hence, $\int_0^n f(x)\, d\alpha(x) = \int_0^n f(x)\, d[x]$ exists

Similarly, and,

$$\int_0^n f(x)\, d[x] = f(0)\alpha_0 + f(1)\alpha_1 + f(2)\alpha_2 + \cdots + f(n)\alpha_n = \sum_{k=1}^{n} f(k)\alpha_k$$

$$= 0\cdot 0 + f(1)\cdot 1 + f(2)\cdot 1 + \cdots + f(n)\cdot 1$$

$$= \sum_{k=1}^{n} f(k)$$

$$\int_0^n f(x)\, d[x] = \sum_{k=1}^{n} a_k$$

Hence the proof.

Theorem: 10

U.S.Q 15M State and prove Euler's Summation Formula:

Statement:

If f has a continuous derivative f' on $[a,b]$, then we have

$$\sum_{a<n\le b} f(n) = \int_a^b f(x)\,dx + \int_a^b f'(x)\,((x))\,dx + f(a)((a)) - f(b)((b))$$

where $((x)) = x - [x]$

When a and b are integers, this becomes

$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)\,dx + \int_a^b f'(x)\,(x - [x] - \tfrac{1}{2})\,dx + \frac{(f(a)+f(b))}{2}$$

Here $a<n\le b$ means the sum from $n = [a]+1$ to $n = [b] = b$

Proof:

By Integration by parts, (theorem 4)

$$\int_a^b f\, d\alpha + \int_a^b \alpha\, df = f(b)\,\alpha(b) - f(a)\,\alpha(a)$$

Let $\alpha(x) = x - [x] = ((x))$ then we have

$$\int_a^b f(x)\, d(x - [x]) + \int_a^b (x - [x])\, df(x) = f(b)\,\alpha(b) - f(a)\,\alpha(a)$$

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$$\Rightarrow \int_a^b f(x)\,d(x-[x]) + \int_a^b (x-[x])\,df(x) = f(b)(b-[b]) - f(a)(a-[a])$$

$$\Rightarrow \int_a^b f(x)\,dx - \int_a^b f(x)\,d[x] + \int_a^b ((x))\,f'(x)\,dx = f(b)((b)) - f(a)((a))$$

$$\Rightarrow \int_a^b f(x)\,d[x] = \int_a^b f(x)\,dx + \int_a^b ((x))\,f'(x)\,dx + f(a)((a)) - f(b)((b)) \longrightarrow ①$$

$$\Rightarrow \int f(x)\,d[x] = \int f(x)\,dx - \dots$$

consider the partition P such that

$$P = \{ a = [a], [a]+1, [a]+2, \dots [b] = b \} \text{ of } [a,b]$$

The jump at 'a' is 0 and the jump at all the other points is 1.

Then, by thm: 9 we have

$$\int_0^n f\,d[x] = \sum_{k=1}^{n} f(k) = \sum_{k=1}^{n} a_k$$

Here, $\displaystyle\int_a^b f\,d[x] = \sum_{k=1}^{n} f(k)\,\alpha_k$

$$= f(a)\cdot 0 + f([a]+1)\cdot 1 + f([a]+2)\cdot 1 + \dots + f([b])\cdot 1$$

$$= \sum_{n=[a]+1}^{[b]} f(n)$$

$$\int_a^b f\,d[x] = \sum_{a<n\le b} f(n) \longrightarrow ②$$

Using ② in ①:

$$\sum_{a<n\le b} f(n) = \int_a^b f(x)\,dx + \int_a^b f'(x)\,((x))\,dx + f(a)((a)) - f(b)((b)) \longrightarrow ③$$

If 'a' and 'b' are integers then

$$((a)) = a - [a] = a - a = 0$$

and $((b)) = b - [b] = b - b = 0$